CS-151 Quantum Computer Science: Problem Set 7

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Guidelines: ***The deadline to return this problem set is 11.59pm on Friday, April 5***. *Remember that you can collaborate with each other in the preliminary stages of your progress, but each of you must write their solutions independently. Submission of the problem set should be via Gradescope only.*

Problem 1 (30 points)**.**

a) *Draw a quantum circuit which performs quantum phase estimation on a unitary matrix U and outputs the eigenvalue corresponding to a specific eigenvector $|u\rangle$, which is provided as an input, with up to $t = 1$ bit of precision.*

b) *Let U be a unitary transformation with eigenvalues ± 1, which acts on a state $|\psi\rangle$. Using the quantum phase estimation procedure, construct a quantum circuit to collapse $|\psi\rangle$ into one or the other of the two eigenspaces of the operator U. Analyze this circuit and explain how this circuit also gives a classical indicator as to which eigenspace the final state is in. (Hint: You can use the circuit from part (a).)*

c) *Let $U = Z^{\otimes n}$ and $|\psi\rangle = (\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle)^{\otimes n}$. Suppose that using the quantum circuit from part (b), we obtain the eigenvalue $+1$. What can we say about the state stored in the $|\psi\rangle$ register? Compute the probability of obtaining $+1$ using this algorithm.*

Problem 2 (30 points)**.** *Let*

$$|u_s\rangle = \frac{1}{\sqrt{r}} \sum_{k=1}^{r-1} e^{-2\pi i s k/r} |x^k \mod N\rangle$$

a) *Prove that $\langle u_s | u_{s'} \rangle = 0$ for $s \neq s'$.*

b) *Let $|1\rangle$ be the ket representing the identity element in the multiplicative group of integers mod N. (If this doesn't mean anything to you, you can just think of it as the integer 1). Show that*

$$|1\rangle = \frac{1}{\sqrt{r}} \sum_{k=1}^{r-1} |u_s\rangle$$

Problem 3 (40 points)**.**

a) *Recall that the expected value of a random variable is just the probability-weighted sum over all possible outcomes.*

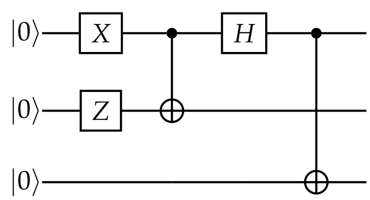
$$\mathbb{E}[X] = \sum_i \Pr(X = x_i) \cdot x_i$$

Measuring *an observable O with respect to a quantum state $|\psi\rangle$ is given by measuring $|\psi\rangle$ in terms of the eigenbasis of O. i.e., Measuring a label i according to a POVM $\{\ |v_i\rangle\langle v_i|\ \}_i$, and outputting the eigenvalues λ_i. Show that the average value of the observable O, written $\overline{O}$, with respect to some state $|\psi\rangle$ is given by,*

$$\overline{O} = \langle\psi|O|\psi\rangle$$

b) *Let $O = X \otimes X \otimes X$ be an observable. Verify that the eigenbasis for O is orthogonal, that is, the eigenvectors of O form an orthogonal basis.*

c) *Let F be the random variable which corresponds to measuring the output of the following circuit according to the observable $O = X \otimes X \otimes X$.*



Give a complete description of F. What is the expectation value of F?

d) *What are the possible outcomes of measuring $X^{\otimes n}$? (Hint: the possible values one would measure from an observable are its eigenvalues.)*